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One-Dimensional Space-Discrete Transport Subject to Lévy Perturbations

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Abstract In this paper we study a one-dimensional space-discrete transport equation subject to additive Lévy forcing. The explicit form of the solutions allows their analytic study. In particular we discuss the invariance of the covariance structure of the stationary distribution for Lévy perturbations with finite second moment. The situation of more general Lévy perturbations lacking the second moment is considered as well. We moreover show that some of the properties of the solutions are pertinent to a discrete system and are not reproduced by its continuous analogue.

Keywords Transport equation · Lévy process · Lévy flights · Bessel function · Stationary distribution

1 Introduction

The standard explanatory approach to properties of fully developed isotropic turbulence developed by Kolmogorov, Richardson, Bachelor and many others [3] is based on the concept of the energy cascade which transports the energy from the larger structures (eddies) of the flow to ever smaller ones. This concept is motivated by the observation that in three dimensional turbulence the energy is typically put into the system through the large scale forcing (e.g. via the global pressure gradient) and is dissipated into heat at the smallest, viscous scale of the system. This kind of explanation is very successful in qualitative interpretation of experimental results, and leads theorists to try to design simpler continuous models possessing at least some properties in common with real turbulent flows in fluids. However, 'even these simple models remain surprisingly complicated to analyse' [7]. This has lead the authors

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of [6, 7] to an even more simplified version of a discrete transport equation, corresponding to a shell model neglecting all nonlinearities with energy injection through stochastic forcing at the largest scale. As they show, even this linear model exhibits behaviour which is highly nontrivial. The corresponding system of coupled linear stochastic differential equations reads after time-integration as

$$\begin{cases} a_0^\nu(t;s) = 0, \\ a_1^\nu(t;s) = a_1^\nu(s) + \int_s^t (-a_2^\nu(u;s) - \nu a_1^\nu(u;s))\,du + L_s(t), \\ a_n^\nu(t;s) = a_n^\nu(s) + \int_s^t (a_{n-1}^\nu(u;s) - a_{n+1}^\nu(u;s) - \nu a_n^\nu(u;s))\,du, \quad n \geq 2,\ t \geq s, \end{cases} \tag{1}$$

with $\nu \geq 0$ and for $-\infty < s \leq t < \infty$. The system (1) is a space-discrete transport equation with a Dirichlet boundary condition and a singular random perturbation at the position $n = 1$. The interpretation of the equations corresponds to the situation when the energy is pumped into the system at shell $n = 1$ via the effective noise (or signal) term $L_s(t)$, is transported between the shells and dissipated due to the effective viscosity $\nu \geq 0$.

Let $L = (L(t))_{t\geq 0}$ be a one dimensional Lévy process, i.e. a stochastically continuous process with stationary independent increments starting at the origin and having right-continuous sample paths with left limits, and let

$$L_s(t) := L(t-s), \quad -\infty < s \leq t, \tag{2}$$

denote a time-shifted Lévy process defined on the half-line $[s, +\infty)$. It is well known that a Lévy process L is completely characterised by the Fourier transform of its marginal distributions which has the following form due to the Lévy–Khinchin formula:

$$\begin{aligned} &\mathbf{E} e^{i\lambda L(t)} = e^{t\Psi(\lambda)}, \quad \lambda \in \mathbb{R},\ t \geq 0, \\ &\Psi(\lambda) = -\sigma^2 \frac{\lambda^2}{2} + i\lambda\mu - \int_{\mathbb{R}\backslash\{0\}} \left(e^{i\lambda y} - 1 - i\lambda \frac{y}{1+y^2} \right) \rho(dy). \end{aligned} \tag{3}$$

The function $\Psi(\cdot)$ is called the characteristic exponent of L. The first summand of Ψ corresponds to a Brownian motion with variance σ, $\sigma \geq 0$, the second summand determines the linear drift μt, $\mu \in \mathbb{R}$, and the third summand corresponds to a pure jump Lévy processes whose jump intensity and sizes are governed by the jump measure ρ satisfying the integrability condition $\int_{\mathbb{R}\backslash\{0\}} (y^2 \wedge 1)\, \rho(dy) < \infty$.

In particular, if $\rho \equiv 0$, the Lévy process L is just a Brownian motion with drift. The case of the standard Brownian motion without drift, corresponding to the characteristic exponent $\Psi(\lambda) = -\lambda^2/2$, is exactly the one discussed in [6]. Further, if $\rho(\mathbb{R}\backslash\{0\}) = c \in (0, \infty)$, and $\sigma = \mu = 0$, then L is a compound Poisson process with intensity c and jump sizes distributed with the probability law $\rho(\cdot)/c$. Finally, the case $\sigma = \mu = 0$ and $\rho(dy) = c(\alpha)|y|^{-1-\alpha} dy$, $\alpha \in (0, 2)$, corresponds to Lévy flights with $\Psi(\lambda) = -|\lambda|^\alpha$. We address the reader to the books [2, 8] for more information on Lévy processes.

The goal of this paper consists in studying the long time behaviour of solutions of the system (1). Our approach here differs from the one of [6] and is based on the explicit solution of (1), which allows a generalisation of the results of [6]. We moreover discuss some peculiar properties of the continuous limit of (1) described by a partial differential equation. The behaviour of the solutions of this continuous equation turns out to be different from the one of its discrete analogue (1).

2 Explicit Solution

Let us first derive the explicit form of the solution of (1).

Proposition 1 *The solution of* (1) *is given by*

$$
\begin{aligned}
a_0^\nu(t;s) &= 0,\\
a_n^\nu(t;s) &= e^{-\nu(t-s)}\sum_{m=1}^{\infty} a_m(s)\big[J_{|n-m|}(2(t-s)) + (-1)^{m-1}J_{n+m}(2(t-s))\big]\\
&\qquad + \int_s^t H_n^\nu(t-r)\,dL_s(r), \quad n\ge 1,\ t\ge s,\\
H_n^\nu(r) &:= n\frac{J_n(2r)}{r}e^{-\nu r},
\end{aligned} \tag{4}
$$

$J_n(\cdot)$ *being the Bessel functions of the first kind.*

Proof For brevity we set $s=0$, $a_n^\nu := a_n^\nu(0)$ and $a_n^\nu(t) := a_n^\nu(t;0)$. Since the system (1) is linear it is enough to solve the homogeneous deterministic system for initial conditions $a_m^\nu = 1$, $m\ge 1$, $a_n^\nu = 0$, $n\ne m$, and the non-homogeneous stochastic system with initial conditions $a_n^\nu = 0$, $n\ge 0$. Thus the solution of the homogeneous system is given by

$$
\begin{aligned}
a_n^\nu(t) &= e^{-\nu t}(J_{|n-m|}(2t) + (-1)^{m-1}J_{n+m}(2t)),\\
a_m^\nu(t) &= e^{-\nu t}(J_0(2t) + (-1)^{m-1}J_{2m}(2t)).
\end{aligned} \tag{5}
$$

The proof is straightforward with the help of the differentiation formulae (here and below, see [1, Chapter 9])

$$
\begin{aligned}
2J_n'(t) &= J_{n-1}(t) - J_{n+1}(t), \quad n\ge 1,\\
J_0'(t) &= -J_1(t).
\end{aligned} \tag{6}
$$

The initial condition is satisfied due to the relations $J_0(0)=1$, $J_n(0)=0$, $n\ge 1$.

To show that $\int_0^t H_n^\nu(t-r)\,dL(r)$ satisfies the non-homogeneous equations we have to be sure that the stochastic integral is well-defined for all $t\ge 0$. First we note that the asymptotic expansion

$$
J_n(2r) \approx \frac{r^n}{n!}, \quad r\to 0,\ n\ge 1, \tag{7}
$$

implies that the integrand is a smooth and bounded function on $r\in[0,\infty)$ for all $\nu\ge 0$. Consequently the stochastic integral is well-defined and we can easily calculate the characteristic function

$$
\mathbf{E}\exp\left(i\lambda\int_0^t H_n^\nu(t-r)\,dL(r)\right) = \exp\left(\int_0^t \Psi(\lambda H_n^\nu(r))\,dr\right), \quad \lambda\in\mathbb{R}. \tag{8}
$$

For the next step of our argument we take use of the relations (6) to show that for $n\ge 1$

$$\begin{aligned}\frac{d}{dr}H_n^\nu(r) &= -n\frac{J_n(2r)}{2r^2}e^{-\nu r} + n\frac{J_{n-1}(2r) - J_{n+1}(2r) - nJ_n(2r)}{2r}e^{-\nu r}\\ &= -\frac{J_{n-1}(2r) + J_{n-1}(2r)}{2s}e^{-\nu s} + n\frac{J_{n-1}(2r) - J_{n+1}(2r) - \nu J_n(2r)}{2r}e^{-\nu r}\\ &= H_{n-1}^\nu(r) - H_{n+1}^\nu(r) - \nu H_n^\nu(r).\end{aligned} \tag{9}$$

Thus we finish the proof with help of the integration by parts. Indeed for $n \geq 1$ we have

$$\begin{aligned}\int_0^t H_n^\nu(t-r)\,dL(r) &= H_n^\nu(0)L(t) + \int_0^t \left[\int_0^u (H_n^\nu)'(t-r)\,dL(r)\right]du\\ &= H_n^\nu(0)L(t) + \int_0^t (a_{n-1}^\nu(u) - a_{n+1}^\nu(u) - \nu a_n^\nu(u))\,du.\end{aligned} \tag{10}$$

Taking into account the expansion in (7) we note that $H_n^\nu(0) = 1$ for $n = 1$ and $H_n^\nu(0) = 0$ for $n \geq 2$, which finishes the proof. □

3 Stationary Distribution

In this section we study the convergence of $a_n^\nu(t, s)$ to the stationary distribution.

It is clear that due to the linearity of the system, the convergence to the stationary regime depends on the asymptotic properties of the deterministic solution corresponding to $L \equiv 0$. In [6], the authors showed that for $(a_n^\nu)_{n\geq 1} \in l^2$, $\nu \geq 0$, the deterministic solution converges weakly to zero. For bounded initial values of a_n, they gave an example of nonconvergence, namely the solution

$$\begin{aligned}&a_{2n-1}^0(t) = 1,\\ &a_{2n}^0(t) = 0, \quad n \geq 1,\end{aligned} \tag{11}$$

which is a fixed point of the homogeneous system. With help of the explicit formula for the solution, we can find another bounded initial condition, for which the corresponding solution weakly converges to zero. Indeed, for $a_{2n-1} = 0$, $a_{2n} = 1$, $n \geq 1$ we have:

$$\begin{aligned}a_{2n-1}^0(t) &= 2\sum_{j=1}^{n-1} J_{2j-1}(2t) + J_{2n-1}(2t),\\ a_{2n}^0(t) &= J_0(2t) + 2\sum_{j=1}^{n-1} J_{2j}(2t) + J_{2n}(2t).\end{aligned} \tag{12}$$

The weak convergence to zero follows from the asymptotic expansion

$$J_n(2t) \approx \sqrt{\frac{1}{\pi t}}\cos\left(2t - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad t \to \infty. \tag{13}$$

The convergence is however not uniform over $n \geq 1$ since for any $t \geq 0$

$$2\sum_{j=1}^{n-1} J_{2j-1}(2t) \to \int_0^{2t} J_0(r)\,dr, \quad n \to \infty, \tag{14}$$

and

$$J_0(2t) + 2\sum_{j=1}^{n-1} J_{2j}(2t) \to 1, \quad n \to \infty. \tag{15}$$

To study the influence of the stochastic forcing, let us set $a_n^\nu(s) := 0$ for $n \ge 0$, so that the deterministic part of the solution disappears. To obtain the invariant distribution, we consider the limit

$$a_n^\nu = \lim_{s\to-\infty} \int_s^0 H_n^\nu(-r)\, dL_s(r) \stackrel{d}{=} \int_0^\infty H_n^\nu(r)\, dL(r), \tag{16}$$

where '$\stackrel{d}{=}$' denotes equality in law. It is instructive to determine the Fourier transform of a_n^ν which can be found as

$$F_n^\nu(\lambda) = \lim_{s\to-\infty} \mathbf{E} e^{i\lambda a_n^\nu(0,s)} = \mathbf{E}\exp\left(\int_0^\infty \Psi(\lambda H_n^\nu(r))\, dr\right), \quad \lambda \in \mathbb{R}. \tag{17}$$

First of all, we have to study the existence of the latter limit. Clearly, a_n^ν is a well-defined random variable if and only if for any finite λ

$$\int_0^\infty \Psi(\lambda H_n^\nu(r))\, dr < \infty. \tag{18}$$

In view of the asymptotics (13) this is equivalent to the convergence of the integral

$$\int_1^\infty \Psi\left(\frac{e^{-\nu r}\cos(2r)}{\sqrt{r}}\right) dr < \infty, \tag{19}$$

which in turn depends on the asymptotics of $\Psi(r)$ as $r \to 0$. We also notice that the condition (19) does not depend on n, thus all random variables a_n^ν are either well-defined or not well-defined simultaneously. In particular, for any $n, m \ge 1$ we can calculate the mutual Fourier transform

$$F_{n,m}^\nu(\lambda_n, \lambda_m) := \mathbf{E} e^{i(\lambda_n a_n^\nu + \lambda_m a_m^\nu)} = \exp\left(-\int_0^\infty \Psi(\lambda_n H_n^\nu(r) + \lambda_m H_m^\nu(r))\, dr\right). \tag{20}$$

Clearly, similar formulae hold for any finite number of summands.

In the next section we study the invariant distributions a_n^ν in more detail by discussing several special situations corresponding to different forms of the characteristic exponent Ψ.

4 Particular Cases

4.1 Linear Perturbation

We start with the simplest deterministic perturbation $L(t) = \mu t$ linear in time (corresponding to a constant forcing) with the characteristic exponent $\Psi(\lambda) = i\lambda\mu$. The Fourier transform of the stationary solution is found explicitly as

$$\begin{aligned} \mathbf{E} e^{i\lambda a_n^\nu} &= \exp\left(\int_0^\infty \Psi(\lambda H_n^\nu(r))\, dr\right) \\ &= \exp\left(i\lambda\mu \int_0^\infty H_n^\nu(r)\, dr\right) \end{aligned}$$

$$= \exp\left(i\lambda\mu\int_0^\infty n\frac{J_n(2r)}{r}e^{-\nu r}\,dr\right)$$
$$= \exp\left(i\lambda\mu n\left(\frac{\nu}{2}+\sqrt{1+\frac{\nu^2}{4}}\right)^{-n}\right). \tag{21}$$

This implies that the stationary solution is a constant

$$a_n^\nu = \mu n\left(\frac{\nu}{2}+\sqrt{1+\frac{\nu^2}{4}}\right)^{-n} \tag{22}$$

and in the limit $\nu \to 0$ we obtain that

$$a_n^\nu \approx \mu n\left(1-\frac{n}{2}\nu+\cdots\right) \to a_n^0 = \mu n. \tag{23}$$

On the other hand as $n \to \infty$, $a_n^\nu \to 0$ for $\nu > 0$, whereas $a_n^0 \to \infty$.

4.2 Gaussian Perturbation

If the forcing is Gaussian, i.e. $\Psi(\lambda) = -\sigma^2\lambda^2/2$, it is clearly seen that each a_n^ν is also a Gaussian random variable with the characteristic function

$$\mathbf{E}e^{i\lambda a_n^\nu} = \exp\left(\int_0^\infty \Psi(\lambda H_n^\nu(r))\,dr\right)$$
$$= \exp\left(-\lambda^2\frac{\sigma^2}{2}\int_0^\infty (H_n^\nu(r))^2\,dr\right)$$
$$= \exp\left(-\lambda^2\frac{\sigma^2}{2}\int_0^\infty n^2\frac{J_n^2(2r)}{r^2}e^{-2\nu r}\,dr\right)$$
$$= \exp\left(-\frac{\lambda^2\sigma^2}{2\sqrt{\pi}}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n)}\nu^{1-2n}{}_3F_2\left(n-\frac{1}{2}, n, n+\frac{1}{2}; n+1, 2n+1; -\frac{4}{\nu^2}\right)\right). \tag{24}$$

In the limit as $\nu \to 0$ we recover

$$\mathbf{E}e^{i\lambda a_n^\nu} \approx \exp\left(-\frac{\lambda^2\sigma^2}{2}\left(\frac{8n^2}{\pi(4n^2-1)}-n\nu+\cdots\right)\right)$$
$$\to \mathbf{E}e^{i\lambda a_n^0} = \exp\left(-\frac{\lambda^2\sigma^2}{\pi}\left(1-\frac{1}{4n^2}\right)^{-1}\right) \tag{25}$$

for all $n \geq 1$. In particular,

$$\mathbf{E}(a_n^0)^2 = \frac{2}{\pi}\sigma^2\left(1-\frac{1}{4n^2}\right)^{-1} \to \frac{2}{\pi}\sigma^2, \quad n\to\infty, \tag{26}$$

as it was discovered in [6]. Taking into account (20), we determine the covariances,

$$\begin{aligned}
&\mathbf{E}a_m^\nu a_n^\nu \\
&\quad = -\frac{\partial^2}{\partial\lambda_m\partial\lambda_n} F_{n,m}^\nu(\lambda_n,\lambda_m)\Big|_{\lambda_n=\lambda_m=0} = \sigma^2 \int_0^\infty H_m^\nu(r) H_n^\nu(r)\,dr \\
&\quad = \sigma^2 (2\nu)^{1-m-n} \frac{(m+n-2)!}{(m-1)!(n-1)!} \\
&\qquad \times {}_4F_3\left(\frac{m+n-1}{2}, \frac{m+n}{2}, \frac{m+n+1}{2}, \frac{m+n}{2}+1; m+1, n+1, m+n+1; -\frac{4}{\nu^2}\right), \\
&\mathbf{E}a_m^0 a_n^0 \\
&\quad = \sigma^2 \frac{2}{\pi} \cos\left(\pi\frac{m-n}{2}\right)\left[\frac{1}{(m+n)^2-1} - \frac{1}{(m-n)^2-1}\right],
\end{aligned} \tag{27}$$

where the latter formula coincides with the result from [6].

4.3 Lévy Perturbations with Finite Second Moment

Further, we note that the covariance structure of the stationary distribution is the same for all Lévy forcings with the finite second moment. Indeed, assume that $\mathbf{E}L(t)^2 < \infty$ which is equivalent to the condition that $\Psi(\cdot) \in C^2(\mathbb{R})$. Differentiating the Fourier transform (20) at zero, we determine the mean value and the covariances of the stationary solutions a_n^ν:

$$\begin{aligned}
\mathbf{E}a_n^\nu &= \frac{d}{d\lambda} F_n^\nu(\lambda)\Big|_{\lambda=0} = -i\Psi'(0)\int_0^\infty H_n^\nu(r)\,dr, \\
\mathbf{E}(a_n^\nu)^2 &= -\frac{d^2}{d\lambda^2} F_n^\nu(\lambda)\Big|_{\lambda=0} = -\Psi''(0)\int_0^\infty (H_n^\nu(r))^2\,dr - \left(\Psi'(0)\int_0^\infty H_n^\nu(r)\,dr\right)^2, \\
\mathbf{E}a_m^\nu a_n^\nu &= -\frac{\partial^2}{\partial\lambda_m\partial\lambda_n} F_{n,m}^\nu(\lambda_n,\lambda_m)\Big|_{\lambda_n=\lambda_m=0} \\
&= -\Psi''(0)\int_0^\infty H_m^\nu(r)H_n^\nu(r)\,dr - (\Psi'(0))^2\int_0^\infty H_m^\nu(r)\,dr\int_0^\infty H_n^\nu(r)\,dr.
\end{aligned} \tag{28}$$

The closed form for the above integrals was already given in (21), (24) and (27). Thus our analysis shows that the covariance structure of the stationary laws does not depend on Gaussianity itself, and is the same for all Lévy forcings with equal first and second moments.

4.4 Lévy Flights

Finally, for the Lévy flights forcing with $\Psi(\lambda) = -\sigma^\alpha|\lambda|^\alpha$, $\alpha \in (0,2)$, $\sigma > 0$, the stationary probability distribution exists for all $n \geq 1$, $\alpha \in (0,2)$, and $\nu > 0$ and for all $\alpha \in (\frac{2}{3}, 2)$ and $\nu = 0$, and is also α-stable with the characteristic function

$$\mathbf{E}e^{i\lambda a_n^\nu} = \exp\left(-|\lambda|^\alpha\sigma^\alpha n^\alpha\int_0^\infty \frac{|J_n(2r)|^\alpha}{r^\alpha} e^{-\alpha\nu r}\,dr\right). \tag{29}$$

In the zero viscosity case $\nu = 0$, we observe a strange transition at the critical value $\alpha = 2/3$ of the stability index. Formally it comes from the integrability condition of the Bessel functions. Indeed, in view of the asymptotics (13) the limits

$$\lim_{t\to\infty}\int_0^t \frac{|J_n(2r)|^\alpha}{r^\alpha}\,dr, \quad \lim_{t\to\infty}\int_1^t \frac{|\cos(t)|^\alpha}{r^{3\alpha/2}}\,dr \quad \text{and} \quad \lim_{t\to\infty}\int_1^t \frac{1}{r^{3\alpha/2}}\,dr \tag{30}$$

either exist or do not exist simultaneously, and the latter limit is finite if only if $\alpha > 2/3$. Since the stability index determines the weight of the tails of the Lévy flights, $\mathbf{P}(|L(t)| > u) = \mathcal{O}(u^{-\alpha})$, $u \to \infty$, we can conclude that in the model under consideration, too heavy tails and big jumps of the random forcing lead to anomalous dispersion of the energy in space and do not allow to build up a stationary *probability* distribution. On the other hand as it will be shown in the next section, the critical value $\alpha = 2/3$ has it origin not in the nature of the random forcing but rather in the properties of the space-discrete deterministic equation.

5 Continuous Limit

In this section we discuss an interesting effect related to the continuous limit of (1). For any real function f we introduce a discrete symmetric gradient at x as

$$\nabla_h f(x) := \frac{f(x+h) - f(x-h)}{2h}, \quad h > 0. \tag{31}$$

Denote also $a^\nu(t, n; s) := a_n^\nu(t; s)$. With this notation under condition $h = 1$, the system (1) takes the form

$$\begin{cases} a^\nu(t, 0; s) = 0, \\ a^\nu(t, h; s) = a(h; s) + \displaystyle\int_s^t (-2\nabla_h a^\nu(u, h; s) - \nu h a^\nu(u, h; s))\,du + \tfrac{1}{h} L_s(th), \\ a^\nu(t, nh; s) = a(nh; s) + \displaystyle\int_s^t (-2\nabla_h a^\nu(u, nh; s) - \nu h a^\nu(u, nh; s))\,du, \quad n \geq 2, \end{cases} \tag{32}$$

∇_h denoting the discrete gradient w.r.t. the second variable of a^ν. We omit the discussion on the limiting behaviour of the deterministic solution assuming that the initial values of a^ν are identically zero. According to (16), we find the stationary solution of the perturbed system as

$$a_h^\nu(n) = \lim_{s\to-\infty} \frac{n}{h}\int_{-s}^0 \frac{J_n(-2r)}{-r} e^{\nu h r}\,dL_s(hr) \overset{d}{=} \frac{n}{h}\int_0^\infty \frac{J_n(2r)}{r} e^{-\nu h r}\,dL(hr), \tag{33}$$

provided the latter stochastic integral exists. To study the existence, we use the change of variables formula

$$\int_0^\infty f(r)\,dL(hr) \overset{d}{=} \int_0^\infty f(\tfrac{r}{h})\,dL(r) \tag{34}$$

for $h > 0$ and a continuous function f to obtain that

$$a_h^\nu(n) \overset{d}{=} \frac{n}{h}\int_0^\infty \frac{J_n(2r)}{r} e^{-\nu h r}\,dL(hr) \overset{d}{=} \frac{n}{h}\int_0^\infty \frac{J_n(\frac{2r}{h})}{\frac{r}{h}} e^{-\nu r}\,dL(r). \tag{35}$$

It is clearly seen, that the stochastic integral on the right-hand side of (35) exists for all $n \geq 1$ and $h > 0$ if and only if the integrability condition (19) holds (recall constraints on α in the

Lévy flights case). Further, let n and h be such that $nh = x > 0$. Then with help of (35) we can pass to the limit

$$\begin{aligned} a_h^\nu(x) &\stackrel{d}{=} \int_0^\infty \frac{x}{h} \frac{J_{\frac{x}{h}}(\frac{x}{h}\cdot\frac{2r}{x})}{r} e^{-\nu r}\,dL(r) \stackrel{h\to 0}{\to} \int_0^\infty \frac{\delta(\frac{2r}{x}-1)}{r} e^{-\nu r}\,dL(r) \\ &= \frac{2}{x} e^{-\nu\frac{x}{2}} \dot{L}\left(\frac{x}{2}\right) =: a^\nu(x). \end{aligned} \tag{36}$$

To derive the latter limit we also used that $(\mu J_\mu(\mu(x+1)))_{\mu>0}$, is an approximating sequence for the Dirac δ-function (see [5]), i.e. that for any appropriate test function f we have

$$\lim_{\mu\to\infty} \int_0^\infty \mu J_\mu(\mu(x+1)) f(x)\,dx = f(0). \tag{37}$$

This approximating system which appears here naturally as a generic property of the model, also proved to be useful in treating relativistic plasma dispersion in the limit of zero magnetic field, see [4]. We emphasise that the limit (36) is pointwise for $x > 0$ and $h \to 0$, and our argument has sense provided the condition (19) holds and the random variables $a_h^\nu(x)$ are well-defined.

As a partial differential equation analogue of (32), let us consider the linear transport equation

$$\begin{cases} b_t^{\nu,\varepsilon}(t,x;s) + 2\nabla b^{\nu,\varepsilon}(t,x;s) + \nu b^{\nu,\varepsilon}(t,x;s) = \dot{L}_s(t)\delta(x-\varepsilon), & t > s, x > 0, \\ b^{\nu,\varepsilon}(s,x;s) = \varphi(x), \quad x \geq 0, \quad \varphi(0) = 0, \\ b^{\nu,\varepsilon}(t,0;s) = 0, \quad t \geq s, \end{cases} \tag{38}$$

where $\nu \geq 0$, $\varepsilon > 0$, δ denotes the Dirac function, and φ is the initial condition. Such a setting allows to capture the random perturbation as a non-homogeneity in the linear transport equation.

Equation (38) can be easily solved explicitly. Its general solution is given by

$$b^{\nu,\varepsilon}(t,x;s) = \Phi(x-2t) + \int_s^t e^{-\nu(t-r)}\delta(2r-2t+x-\varepsilon)\,dL_s(r) \tag{39}$$

for an arbitrary $\Phi(\cdot)$. The first boundary condition yields:

$$\Phi(u) = \varphi(u+2s), \quad u \geq -2s. \tag{40}$$

The second boundary condition yields:

$$\begin{aligned} \Phi(-2t) &= -\int_s^t e^{-\nu(t-r)}\delta(2r-2t-\varepsilon)\,dL_s(r), \quad t \geq s, \\ \Phi(u) &= -\int_s^{-\frac{u}{2}} e^{\nu(\frac{u}{2}+r)}\delta(2r+u-\varepsilon)\,dL_s(r), \quad u \leq -2s. \end{aligned} \tag{41}$$

Consequently,

$$b^{\nu,\varepsilon}(t,x;s) = \varphi(x-2t+2s) + \int_s^t e^{-\nu(t-r)}\delta(2r-2t+x-\varepsilon)\,dL_s(r), \quad s \leq t \leq s+x/2, \tag{42}$$

and

$$b^{\nu,\varepsilon}(t,x;s)=\int_s^t e^{-\nu(t-r)}\delta(2r-2t+x)\,dL(r-s)$$
$$-\int_s^{t-\frac{x}{2}} e^{-\nu(t-r-\frac{x}{2})}\delta(2r-2t+x-\varepsilon)\,dL_s(r),\quad t\geq s+x/2. \tag{43}$$

In particular, for $t=0$, fixed $x>0$ and $s\leq -x/2$ we have

$$b^{\nu,\varepsilon}(t,x;s)=\int_s^0 e^{\nu r}\delta(2r+x-\varepsilon)\,dL_s(r)-\int_s^{-\frac{x}{2}} e^{\nu(r+\frac{x}{2})}\delta(2r+x-\varepsilon)\,dL_s(r)$$
$$=e^{-\nu\frac{x-\varepsilon}{2}}\dot{L}(-\tfrac{x-\varepsilon}{2})\overset{\varepsilon\to 0}{\to} e^{-\nu\frac{x}{2}}\dot{L}(-\tfrac{x}{2})=:b^\nu(x). \tag{44}$$

This yields the stationary solution for the limiting equation for $\nu\geq 0$:

$$b^\nu(x)\overset{d}{=}e^{-\nu\frac{x}{2}}\dot{L}\left(\frac{x}{2}\right),\quad x>0. \tag{45}$$

It is important to notice that in this case we have no constraints on the forcing L.

As we see, the stationary distributions of $a^\nu(x)$ and $b^\nu(x)$ are different due to the pre-factor $\frac{2}{x}$. However, for $\nu>0$ the influence of this pre-exponential term is not crucial at least for large values of x. Another difference concerns the fact that the asymptotic solution a^ν can be obtained only for a class of forcings satisfying condition (19), whereas the solution b^ν is defined for all Lévy drivers L. The reason for these differences can be the inaccuracy of the approximation of the discrete gradient ∇_h by gradient operator which neglects higher order derivatives. Due to the presence of higher derivatives in ∇_h, oscillating solutions of the deterministic system arise in a discrete system. These lead to complex interference between the modes in the discrete model. This interference makes a^ν to be typically smaller in amplitude than the solution of the continuous model where no such complex interferences arise. Moreover, the interference between many modes in the case of heavy-tailed forcings leads to the divergences and implies non-existence of the corresponding stationary probability distributions. Technically, the reason for this can be explained as follows. The random perturbation comes in as a stochastic integral of an oscillating kernel $J_n(2r)/r$ which under certain conditions can be approximated by the δ-function. However the asymptotics of this kernel at infinity imposes restrictions on the space of admissible test functions, as it happens in the case of zero viscosity and Lévy perturbations with heavy tails and, i.e. when $\nu=0$ and $\mathbf{P}(|L(t)|>u)=\mathcal{O}(u^{-\alpha})$, $u\to\infty$, $\alpha\in(0,2/3]$.

6 Conclusions

In this paper we studied a space-discrete transport equation subject to additive Lévy forcing. We derived an explicit solution of this equation in terms of the Bessel functions of the first kind, and showed that the random forcing comes into solution via a stochastic integral of a certain oscillating kernel. We generalised the results by [6] on the covariance structure of the stationary distribution from the purely Gaussian to the general Lévy case, and in particular established that the covariances are identical for all Lévy forcings with equal first and second moments. In case of zero viscosity and non-Gaussian Lévy flights forcing, we found that the stationary probability distribution exists only for stability indices $\alpha\in$

$(2/3, 2)$. The explanation to this phenomenon is given by comparison of a space-time limit of the space-discrete transport equation with its partial differential equation analogue. An interesting approximating sequence for a Dirac δ-function appears as a natural part of the space-discrete system. In particular, this approximating sequence admits only test functions which have tails lighter than $u^{-2/3}$. The discrepancy between the space-discrete and the PDE models comes from taking into account higher derivatives while considering a space-discrete gradient operator.

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